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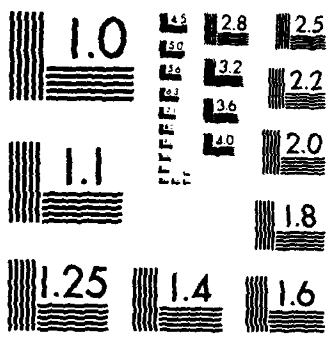
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INTERACTION OF DIFFUSION AND BOUNDARY CONDITIONS

by Jack K. Hale and Carlos Rocha

ABSTRACT

For systems of reaction diffusions, the existence and behavior of the solutions on the compact attractor is discussed for large diffusion coefficients and boundary conditions which can vary from Neumann to Dirichlet conditions

1. Introduction

Consider the system of parabolic partial differential equations (PDE)

$$\partial u/\partial t = D\Delta u + f(u), \quad x \in \Omega \tag{1.1}$$

$$D\partial u/\partial n + \theta E(x)u = 0, \quad x \in \partial\Omega \tag{1.2}$$

where $u \in R^N$, $\Omega \subset R^n$, $n \leq 3$, is a bounded open set with $\partial\Omega$ smooth, $D = \mathrm{diag}(d_1,\ldots,d_N)$, $E = \mathrm{diag}(e_1,\ldots,e_N)$, each $d_j > 0$ is constant, e_j: $\partial\Omega \to R$ is continuous, $e_j > 0$, $j = 1,2,\ldots,n$, and $\theta \in [0,\infty)$ is constant. The function f: $R^N \to R^N$ is supposed to be a $C^{1,1}$-function; that is, continuous and has a Lipschitz continuous first derivative.

An interesting problem is the following one: for fixed functions (f,E), discuss how the flow defined by (1.1), (1.2) depends upon the parameters (D,θ). In a vague sense, the (D,θ)-space should consist of two distinct types of points - those for which the basic structure of the flow does not change significantly when one makes a small change in (D,θ) (the structurally stable points) and those points for which a small change leads to a change in the basic structure of the flow (the bifurcation points).

The purpose of this paper is to make a modest contribution to understanding some parts of this problem. More specifically, we shall give some conditions on (f,E) which will ensure that there is a $d_0 > 0$ such that, for any $d \geq d_0$, $d = \min\{d_j$, $j = 1,2,\ldots,N\}$ and any θ $[0,\infty)$, there is a compact attractor $\mathcal{B}_{D,\theta}$ of (1.1), (1.2) which is upper semicontinuous in D,θ uniformly for $d \geq d_0$, $\theta \geq 0$. Furthermore, $\mathcal{B}_{D,\theta}$ is a singleton for $\theta \geq \theta_0$, θ_0 sufficiently large and converges to an attractor for the Dirichlet problem for (1.1). These results complement the ones obtained previously in

the paper of Hale and Rocha [7], in which they proved the existence and upper semicontinuity of $B_{D,\theta}$ for $d \geq d_0$ and θ in a compact set. The new contribution is the uniformity in $\theta \geq 0$, which permits one to go from Neumann boundary conditions to Dirichlet boundary conditions for any $d \geq d_0$.

For a scalar equation in one-dimension and $d \geq d_0$, the types of bifurcations that occur as one goes from Neumann to Dirichlet conditions is also discussed. There is some overlap in this example with the work of Conley and Smoller [2].

The second aspect of the paper deals with the classification of points in (D,θ)-space as structurally stable or bifurcation points. In this case, we attempt in Section 3 a classification for a scalar one-dimensional equation with f a cubic. These results overlap the ones of Gardner [5] in a special case. The proof of the classification relies heavily upon the transversality theory of Henry [11].

To describe the abstract results more precisely, we need some terminology. Let $X = L^2(\Omega,R^N)$ and define the operator $A = A_{D,\theta}$: $D(A) \to X$ by $A\phi = D\Delta\phi$, where

$$D(A) = \{u \in W^{2,2}(\Omega,R^N): \text{ u satisfies the boundary conditions of (1.2)}\}.$$

Then A is a sectorial operator and one can define the fractional powers A^α of A, $0 \leq \alpha$, and the space $X^\alpha = D(A^\alpha)$ with the graph norm. If $n = 2$, or 3, $n/4 < \alpha < 1$, then $X^\alpha \subset W^{1,2}(\Omega,R^N) \cap L^\infty(\Omega,R^N)$ with continuous inclusion. If $n = 1$, $\alpha = 1/2$, then $X^{1/2} = W^{1,2}(\Omega,R) \cap C(\Omega,R)$. We assume below that α is always chosen in this way. One can then show that (1.1), (1.2) defines a local $C^{1,1}$ semigroup $T_{D,\theta}(t)$ on X^α (see, for example, Henry [10,p.75]).

For any set $B \subset X^\alpha$, the ω-limit set $\omega(B)$ is defined as $\omega(B) = \cap_{\tau \geq 0} Cl \cup_{t \geq \tau} T_{D,\theta}(t)B$. A set $B \subset X^\alpha$ is said to be invariant if, for any ϕ B, one can define

$T_{D,\theta}(t)\phi$ for $t \in R$ and $T_{D,\theta}(t)\phi \in B$ for $t \in R$. A set $\mathcal{A} \subset X^{\alpha}$ is said to be a compact attractor for (1.1), (1.2) if $\mathcal{A}$ is compact, invariant and there is a neighborhood B of $\mathcal{A}$ such that $\omega(B)$ $\mathcal{A}$.

Let $\lambda_j(D,\theta)$ be the first eigenvalues of $-d_j\Delta$ with boundary conditions $d_j\partial u/\partial n + \theta e_j u = 0$, let $\phi_j(D,\theta)$ be the corresponding unit eigenfunctions, $j = 1,2,\ldots,N$, and $\bar{\Phi}_j(D,\theta)$ be the N-dimensional column vector with $\phi_j(D,\theta)$ in the j^{th} place and zero otherwise and let

$$\Lambda_{D,\theta} = \mathrm{diag}(\lambda_1(D,\theta),\ldots,\lambda_N(D,\theta)),$$

$$\Phi_{D,\theta} = (\bar{\Phi}_1(D,\theta),\ldots,\bar{\Phi}_N(D,\theta)).$$

The following ordinary differential equation (ODE), corresponding to the Galerkin approximation obtained by projection of (1.1), (1.2) onto the N-dimensional subspace U spanned by the elements of $\Phi_{D,\theta}$, plays a fundamental role:

$$dv/dt = -\Lambda_{D,\theta}\, v + \int_{\Omega} \Phi_{D,\theta}\; f(\Phi_{D,\theta} v) \tag{1.3}$$

For any set B in R^N, we let $B^U = \{\Phi_{D,\theta} v\colon v \in B^N\}$. For any sets B, C in X^{α}, we let

$$\delta(B,C) = \sup_{x \in B} \mathrm{dist}(x,C)$$

A function $g(\lambda)$ from $\lambda \subset R^k$ to subsets of X^α is said to be upper semicontinuous at λ_0 if $\lim_{\lambda \to \lambda_0} \delta(g(\lambda), g(\lambda_0)) = 0$. For any set $B \subset X^\alpha$, $\epsilon > 0$, let $N(\epsilon, B)$ be the ϵ-neighborhood of B. Let X_0^α designate the fractional power space obtained by taking Dirichlet conditions for (1.1) and $N_0(\epsilon, B)$ be the ϵ-neighborhood of a set $B \subset X_0^\alpha$. Our prinicipal result is

Theorem 1.1. Let $d = \min(d_1, \dots, d_N)$. Suppose there is a compact set $K \subset R^N$ and positive constants $d_0 > 0$, $\delta_0 > 0$ such that the ODE (1.3) has a compact attractor $A_{D,\theta} \subset K$ and $\omega(N(\delta_0, A_{D,\theta})) \subset A_{D,\theta}$ for each $d \geq d_0$, $\theta \geq 0$. Then, for any $0 < \delta_1 < \delta_0$ and any $\epsilon > 0$, there is a $\bar{d}_0 \geq d_0 > 0$ and a compact set $K' \subset L^\infty$ such that (1.1), (1.2) has a compact attractor $B_{D,\theta} \subset K' \cap N(\epsilon, A^U_{D,\theta})$, $\omega(N(\delta_1, A^U_{D,\theta}) \subset B_{D,\theta}$ for $d \geq \bar{d}_0$, $\theta \geq 0$. The attractor $B_{D,\theta}$ is upper semicontinuous in D, θ. Also there is a $\theta_0 > 0$ such that $B_{D,\theta}$ is a singleton $\psi_{D,\theta}$ for $d \geq \bar{d}_0, \theta \geq \theta_0$, $\psi_{D,\theta} \to \psi_{D,\infty}$ as $\theta \to \infty$, $\psi_{D,\infty}$ is a solution of the Dirichlet problem for (1.1) and $\omega(N_0(\delta_1, \psi_{D,\infty})) = \{\psi_{D,\infty}\}$.

This theorem is proved in Section 2.

It is worthwhile to discuss the ideas that are needed to verify the hypothesis in Theorem 1.1. Suppose firstly that $N = 1$, $\Omega = (0,1)$, and $f(u)$ is a polynomial of degree $2p+1$ with $uf(u) \to -\infty$ as $|u| \to \infty$; that is,

$$f(u) = b_{00}u^{2p+1} + b_1 u^{2p} + \dots + b_{2p+1}$$

with $b_0 < 0$. For $N = 1$, the boundary condition (1.2) is equivalent to

$$du_x - \theta\beta_0 u = 0 \qquad \text{at} \qquad x = 0$$

$$du_x - \theta\beta_1 u = 0 \qquad \text{at} \qquad x = 1$$

with $\beta_0 > 0$, $\beta_1 > 0$. The first eigenvalue λ of $-dd^2/dx^2$ with these boundary conditions satisfies $0 \leq \lambda < d\pi^2$ and the corresponding eigenfunction $\phi = \phi(d, \theta)$ can be taken to be

positive. Thus,

$$\tilde{f}(v) \overset{\text{def}}{=} \int_0^1 \phi(x)f(\phi(x)v)dx = \sum_{j=0}^{2p+1} b_{2p+1-j}\left(\int_0^1 \phi^{j+1}(x)dx\right)u^j$$

so that the signs of the coefficients in this polynomial are the same as the ones for f. Also, there are constants $k > 0$, $\delta > 0$ such that $\int_0^1 \phi^{j+1} \leq k$, $j = 0,1,\ldots, 2p+1$ and $\int_0^1 \phi^{2p+2} \geq \delta$ for $d > 0$, $\theta \geq 0$. So (1.3) becomes

(1.4) $\qquad dv/dt = -\theta\lambda v + \tilde{f}(v).$

Since $\theta\lambda_1 \geq 0$ and $\tilde{f}(v)/v^3 \to (\int_0^1 \phi^{2p+2})b_0 \leq \delta b_0 < 0$, it follows that (1.4) has a compact attractor for every $d > 0$, $\theta \geq 0$ and these attractors lie in a compact set. The hypotheses of Theorem 1.1 are satisfied. It is clear that similar conclusions could be drawn for a more general f if the behavior of f is appropriate.

If $N > 1$, the hypotheses are not as easy to verify. For $\Omega = (0,1)$, equations (1.3) in component form are given as

$$dv_j/dt = -\theta\lambda_j v_j + \int_0^1 \phi_j(x)f_j(\phi_1(x)v_1,\ldots,\phi_N(x)v_N)\,dx,\ j = 1,2,\ldots,N$$

Even though all ϕ_j are positive on (0,1), one must assess their relative contributions to the behavior of the flow near $v = \infty$. If all diffusion coefficients are equal, then $\lambda_1 = \ldots = \lambda_N$, $\phi_1 = \ldots = \phi_N$ and the situation is much simpler. Although this topic clearly needs to be investigated in more detail, it will not be pursued in this paper.

2. Proof of Theorem 1.1.

For notational convenience, we take $N = 1$, pointing out in the appropriate places the changes that are needed for $N > 1$. Also, let us first assume $n = 1$, $\Omega = (0,1)$, so that (1.1), (1.2) become

$$(2.1) \qquad u_t = du_{xx} + f(u) \qquad 0 < x < 1$$

$$(2.2) \qquad \begin{aligned} &du_x - \theta\beta_0 u = 0 \qquad \text{at} \quad x = 0 \\ &du_x + \theta\beta_1 u = 0 \qquad \text{at} \quad x = 1 \end{aligned}$$

where β_0, β_1 are given positive constants and $\theta \in [0,\infty)$. If $H^2 = W^{2,2}(\Omega,R)$, $H^1 = W^{1,2}(\Omega,R)$,

$$D(A_{d,\theta}) = \{u \in H^2 : u \text{ satisfies } (2.2)\}$$

$$A_{d,\theta} = -du_{xx},$$

then $A_{d,\theta}$ can be extended as a selfadjoint operator in H^1

$$\int_0^1 (A_{d,\theta} u)v = d\int_0^1 u_x v_x + \theta[\beta_0 u(0)v(0) + \beta_1 u(1)v(1)]$$

defined for every $u,v \in H^1$. Now if we consider the fractional power space $X^{1/2}$ defined as

$$X^{1/2} = D(A_{d,\theta} + I)^{1/2}$$

with the graph norm (see Henry [10, pg. 29]), we have:

$$||(A_{d,\theta}+I)^{1/2}u||_{L^2}^2 = \int_0^1 u(A_{d,\theta}+I)u = \int_0^1 (A_{d,\theta}u)u + \int_0^1 u^2$$

$$= d\int_0^1 u_x^2 + \int_0^1 u^2 + \theta[\beta_0 u^2(0) + \beta_1 u^2(1)]$$

Using the Sobolev inequality $u^2(x) \leq k\ ||u||_{H^1}^2$, we have

$$k_1||u||_{H^1} \leq ||(A_{d,\theta}+I)^{1/2}u||_{L^2} \leq M||u||_{H^1}$$

where $k_1 = \min(d,1), M = [d + \theta(\beta_0 + \beta_1)k]^{1/2}$. Since $X = L^2$ and

$$||(A_{d,\theta}+I)^{1/2}u||_X = ||u||_{X^{1/2}}$$

we conclude that $X^{1/2} = H^1$ independently of θ (Henry [10, p. 167, exercise 10]). Notice though that the constant M in the norm equivalence grows with θ, being unbounded.

We will now consider the eigenvalues and eigenfunctions of $A_{d,\theta}$ and establish uniform estimates for the eigenvalues. If we let $A_{d,\theta}\phi = \lambda\phi$ and $\lambda = d\gamma^2$, then the eigenfunctions and eigenvalues are

$$\lambda_i = d\gamma_i^2;\ \phi_i(x) = d\gamma_i \cos\gamma_i x + \theta\beta_0 \sin\gamma_i x$$

where γ_i are the positive solutions of

$$\text{cotg}\ \gamma = G(\gamma/z)$$

$$G(s) = k(s - s^{-1}),\ k = (\beta_0\beta_1)^{1/2}(\beta_0+\beta_1)^{-1},\ z = \theta d^{-1}(\beta_0\ \beta_1)^{1/2}.$$

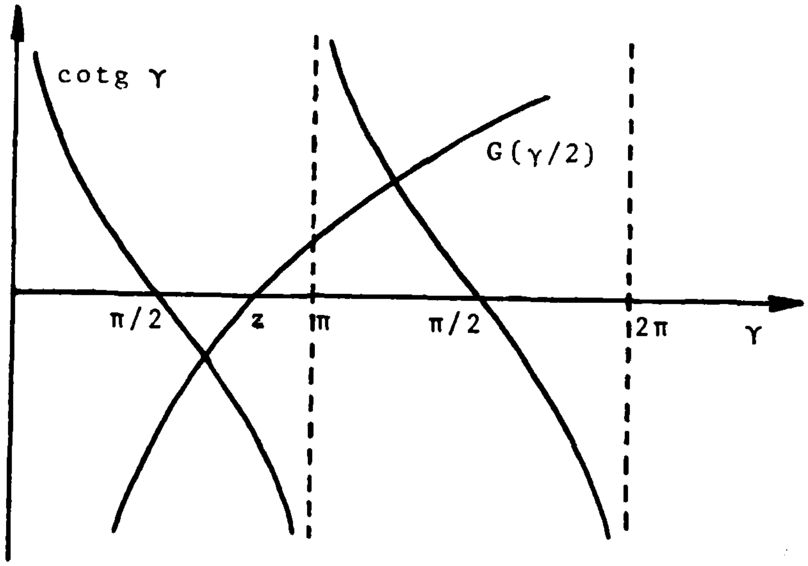

Figure 1

From this, we immediately obtain the estimates

$$\lambda_1 \in (0, d\pi^2);\ \ \lambda_j > d\pi^2 \ \text{ for } \ j \geq 2.$$

Also, from $\gamma = (\lambda/d)^{1/2}$ and $\lim_{d\to\infty} d^{-1/2}\cot(\lambda/d)^{1/2} = \lambda^{-1/2}$, we have

$$\lim_{d\to\infty} \lambda_1 = \theta(\beta_0+\beta_1)$$

Next, we estimate $|\gamma_2 - \gamma_1|$. If $z = \pi$, then

$$\left.\frac{dG(\gamma/\pi)}{d\gamma}\right|_{\gamma=\pi} = \frac{2k}{\pi} \leqslant \frac{1}{\pi}$$

since $2k \leqslant 1$.

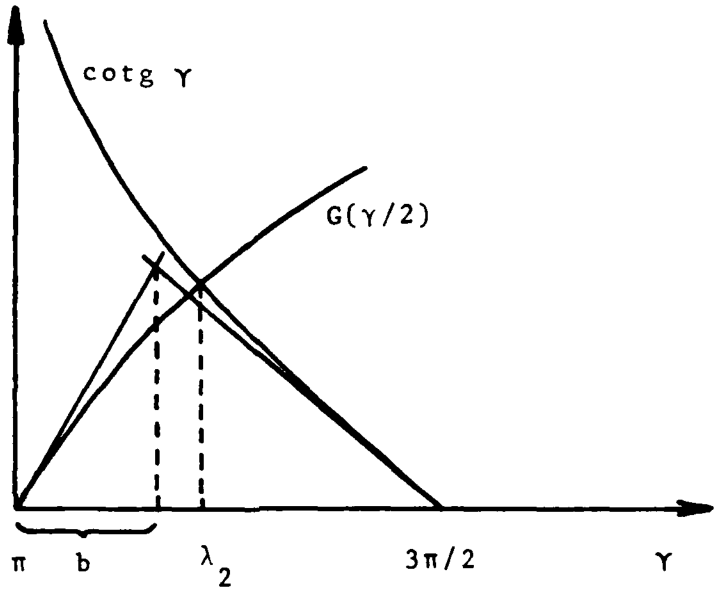

Figure 2

This implies $\gamma_2 - \gamma_1 > b$ where $b = \pi^2/2(\pi+1) > 1$ when $z = \pi$ (see Fig. 2). By continuity, there are $\epsilon > 0$, $\delta > 0$ such that $\gamma_2 - \gamma_1 > \delta$ for $z \in [\pi-\epsilon, \pi+\epsilon]$. If $|z-\pi| > \epsilon$, then $\gamma_2 - \gamma_1 > \epsilon$. Therefore, there is a constant $c > 0$ such that $\gamma_2 - \gamma_1 > c^{1/2}$ for all z. This implies $\lambda_2 - \lambda_1 > dc$ for all $d > 0$, $\theta \in [0, \infty)$.

Now, as in Hale [8] or Hale and Rocha [7], we consider the decomposition $X^{1/2} = Y \oplus Y^{\perp}$, where $Y = \operatorname{span} \phi_1$, and let $T(t)$ denote the semigroup generated by $A_{d,\theta}|Y^{\perp}$. Equations (2.1), (2.2) can be rewritten as

$$\dot{v} = -\lambda_1 v + \tilde{f}(v) + P(v,w)$$

(2.3)

$$w(t,\cdot) = T\ t)w_0 + \int_0^t T(t-s)\ Q(v(s),w(s,\cdot))ds$$

where

$$P(v,w) = \int_0^1 \phi_1[f(v\phi_1 + w) - f(v\phi_1)]$$

(2.4)

$$Q(v,w) = f(v\phi_1 + w) - \phi_1 \int_0^1 \phi_1 f(v\phi_1 + w)$$

$$\tilde{f}(v) = \int_0^1 \phi_1 f(\phi_1 v).$$

For $d > d_0$, $\theta \geq \theta_0$, the assumptions of the theorem imply that the ODE

(2.5) $$\dot{v} = -\lambda_1 v + \tilde{f}(v)$$

has a compact attractor $A_{d,\theta}$ and it attracts a δ_0-neighborhood $N(\delta_0, A_{d,\theta})$ of $A_{d,\theta}$. This implies there is a positively invariant open interval V containing $A_{d,\theta}$ and $A_{d,\theta}$ attracts V. In the case $N > 1$, one uses the converse theorems of Liapunov as in [8] to obtain a positive invariant open set containing $A_{d,\theta}$ and which is attracted by $A_{d,\theta}$.

For each fixed θ and $d \geq d_0$, it was shown in [7] that equation (2.3) has a local integral manifold $w = h(v,d,\theta)$ in a neighborhood of $A^U_{d,\theta}$.

From the uniform estimates of the eigenvalue λ_2, we have, for $w \in Y^{\perp}$,

$$|T(t)w|_{X^{1/2}} \leq k' t^{-1/2} e^{-dct} |w|_X$$

$$|T(t)w|_{X^{1/2}} \leq k' e^{-dct} |w|_X$$

where k' is independent of d, θ. The proof in [7] then shows the existence of $d_1 \geq d_0$ such that $B_{d,\theta} \subset N(\epsilon, A^U_{d,\theta})$, $\omega(N(\delta_1, A^U_{d,\theta})) \subset B_{d,\theta}$ for $d \geq d_1$, $\theta \geq 0$ provided that we know $|\phi|_{L^\infty} \leq k|\phi|_{X^{1/2}}$ for any $\phi \in X^{1/2}$, where k is independent of θ. Next, we establish the continuous inclusion $X^{1/2} \subset L^\infty$ uniformly in θ following Henry [10]. From the Nirenberg-Gagliardo inequality (Henry [10,pg. 37]),

$$\|u\|_{C^0} \leq C \|u\|^{\beta}_{H^2} \|u\|^{1-\beta}_{L^2}$$

for $\beta > 1/4$, and by exercise 11, page 28 of Henry [10], we have $X^\alpha \subset C^0$ $(\alpha > \beta)$ continuously if

$$\|u\|_{C^0} \leq C_1 \|(A_{d,\theta} + I)u\|^{\beta}_{L^2} \|u\|^{1-\beta}_{L^2}.$$

Thus, we need the following estimate, uniform in θ:

$$\|u\|_{H^2} \leq K \|(A_{d,\theta} + I)u\|_{L^2}.$$

If $g = (A_{d,\theta} + I)u$, then we can compute explicitly u as a function of g. In fact,

$$du_{xx} - u = -g$$

implies

$$u(x) = u(0)\ \mathrm{ch}\ \frac{x}{\sqrt{d}} + u_x(0)\sqrt{d}\ \mathrm{sh}\ \frac{x}{\sqrt{d}} - \frac{1}{\sqrt{d}} \int_0^x g(s)\ \mathrm{sh}\ \frac{x-s}{\sqrt{d}}\, ds$$

$$u_x(x) = u(0)\frac{1}{\sqrt{d}}\ \mathrm{sh}\ \frac{x}{\sqrt{d}} + u_x(0)\ \mathrm{ch}\ \frac{x}{\sqrt{d}} - \frac{1}{d} \int_0^x g(s)\ \mathrm{ch}\ \frac{x-s}{\sqrt{d}}\ ds$$

From the boundary conditions,

$$du_x(0) = \theta\, \beta_0\, u(0), \quad du_x(1) = -\theta\, \beta_1 u(1),$$

we have

$$u(1) = u(0)\left[\ \mathrm{ch}\frac{1}{\sqrt{d}} + \theta\, \beta_0 \frac{1}{\sqrt{d}}\ \mathrm{sh}\frac{1}{\sqrt{d}}\right] - \frac{1}{\sqrt{d}}\int_0^1 g(s)\ \mathrm{sh}\frac{1-s}{\sqrt{d}}\, ds$$

$$u_x(1) = u(0)\frac{1}{\sqrt{d}}\left[\mathrm{sh}\frac{1}{\sqrt{d}} + \theta\, \beta_0 \frac{1}{\sqrt{d}}\ \mathrm{ch}\frac{1}{\sqrt{d}}\right] - \frac{1}{d}\int_0^1 g(s)\ \mathrm{ch}\frac{1-s}{\sqrt{d}}\ ds.$$

Hence,

$$u(0) = \left[\mathrm{ch}\frac{1}{\sqrt{d}}\ \theta(\beta_0 + \beta_1) + \frac{1}{\sqrt{d}}\mathrm{sh}\ \frac{1}{\sqrt{d}}(d + \theta^2 \beta_0 \beta_1)\right]^{-1} \int_0^1 g(s)\left[\mathrm{ch}\frac{1-s}{\sqrt{d}} + \theta\, \beta_1 \frac{1}{\sqrt{d}}\ \mathrm{sh}\ \frac{1-s}{\sqrt{d}}\right] ds$$

and

$$u(x) = \frac{\operatorname{ch}\frac{x}{\sqrt{d}} + \theta\beta_0 \frac{1}{\sqrt{d}} \operatorname{sh}\frac{1}{\sqrt{d}}}{(d+\theta^2\beta_0\beta_1)\frac{1}{\sqrt{d}}\operatorname{sh}\frac{1}{\sqrt{d}} + \theta(\beta_0+\beta_1)\operatorname{ch}\frac{1}{\sqrt{d}}} \cdot \int_0^1 g(s)\left[\operatorname{ch}\frac{1-s}{\sqrt{d}} + \theta\beta_1\frac{1}{\sqrt{d}}\operatorname{sh}\frac{1-s}{\sqrt{d}}\right]ds$$

$$-\frac{1}{\sqrt{d}}\int_0^x g(s)\operatorname{sh}\frac{x-s}{\sqrt{d}}\,ds.$$

So finally we obtain $\|u\|_{L^2} \leq R(\theta)\|g\|_{L^2}$ where $R(\theta)$ is a rational function of θ such that for some constant $\bar{K}$, $R(\theta) \leq \bar{K}$ for every $\theta \geq 0$. Also $\|du_{xx}\|_{L^2} = \|u-g\|_{L^2} \leq \|u\|_{L^2} + \|g\|_{L^2}$ and we easily obtain:

$$\|u\|_{H^2} \leq K\|g\|_{L^2} = K\|(A_{d,\theta} + I)u\|_{L^2}.$$

This gives the embedding of $X^{1/2}$ into L^∞ uniform in θ.

Our next objective is to show that $B_{d,\theta}$ is a singleton if $d \geq d_1(r)$, $\theta \geq \theta_0(r)$ with $d_1(r)$, $\theta_0(r)$ sufficiently large. To do this, one uses the following fact: for any $r > 0$, there are $d_1 > 0$, $\theta_0 > 0$ such that $\lambda_1(d,\theta) > r$ for all $d > d_1$, $\theta > \theta_0$. Since $\mathcal{A}_{d,\theta} \subset K$, a compact set for all $d \geq d_0$, $\theta \geq 0$ and $\mathcal{A}_{d,\theta}$ attracts $N(\delta_0, \mathcal{A}_{d,\theta})$, there are constants k_1, k_2 such that

$$|\tilde{f}(v)| \leq k_1,\ |\tilde{f}'(v)| \leq k_2 \quad \text{for } v \quad \bigcup_{d \geq d_0, \theta \geq 0} \gamma^+(N(\delta_0, \mathcal{A}_{d,\theta}))$$

where γ^+ designates the positive orbit. If v_1, v_2 be two solutions of (2.5), then $z = v_1 - v_2$ satisfies

(2.6) $\quad dz/dt = -\lambda_1 z + \tilde{f}'(v_1(t) + \zeta(t))z$

for some $\zeta(t)$ and $|\tilde{f}'(v_1+\zeta)| \leq k_2$. Thus, if $\lambda_1(d,\theta) > r > k_2$ for $d > d_1(r)$, $\theta > \theta_0(r)$, then $z(t) \to 0$ as $t \to \infty$ exponentially. This implies that the solutions of (2.5) approach an equilibrium $v_0(d,\theta)$ as $t \to \infty$. By the same type of argument, the solution $v_0(d,\theta)$ is hyperbolic. Thus, $A_{d,\theta}$ is a singleton $\{v_0(d,\theta)\}$ and it is hyperbolic with exponent $\geq r-k_2$. This implies that $B_{d,\theta}$ is a singleton $\{\psi_{d,\theta}\}$ which is a hyperbolic equilibrium point and attracts $N(\delta_1,\psi_{d,\theta})$.

There is a constant k_3 such that $|f(u)| \leq k_3$, for $u \in \bigcup_{d \geq d_1(r), \theta_0 \geq \theta(r)} \gamma^+ N(\delta_1,\psi_{d,\theta})$. Since $\psi_{d,\theta}$ is an equilibrium point, it follows that $\partial^2\psi_{d,\theta}(x)/\partial x^2$ is uniformly bounded by k_3. Thus, the set $\{\psi_{d,\theta}, d \geq d_1(r), \theta \geq \theta_0(r)\}$ belongs to a compact set K_r. Let $\theta_j \to \infty$ as $j \to \infty$ be a sequence so that $\psi_{d,\theta_j} \to \psi_{d,\infty}$ as $j \to \infty$. Then, $\psi_{d,\infty}$ is an equilibrium solution of (2.1) satisfying the Dirichlet boundary conditions. This equilibrium is hyperbolic and therefore attracts a neighborhood of itself exponentially. This neighborhood can be chosen in such a way as to attract every limit point of $\{\psi_{d,\theta}, d \geq d_1(r), \theta \geq \theta_0(r)\}$. But this will imply there is only one limit point $\psi_{d,\infty}$ and completes the proof of the theorem for $N = 1, n = 1$.

For $N > 1, n = 1$, the last part of the proof follows in essentially the same way since one can construct a quadratic Liapunov function for (2.6).

The case $n = 2,3$ and arbitrary N follows along the same lines as before. One must obtain good estimates on the first and second eigenvalues of $-\Delta$ and the embedding $X^\alpha \subset L^\infty$ must be uniform in θ. Because of this last fact, all solutions in the attractor can be considered in L^∞. for $\theta \geq \theta_0(r)$, there will be a compact set $K_r \subset L^\infty$ which contains the set $\{\psi_{d,\theta}, d \geq d_1(r), \theta \geq \theta_0(r)\}$ and $\psi_{d,\theta} \to \psi_{d,\infty}$ as $\theta \to \infty$. The function $\psi_{d,\infty}$ will satisfy the Dirichlet problem and regularity theory implies it is in $X_0^\alpha(\Omega,R^N)$. Therefore, we only discuss the second eigenvalue and the uniform embedding of X^α into L^∞. We also only give the proof for $n = 3$ since obvious

changes will give a proof for $n = 2$.

First, we establish uniform bounds in θ for the eigenvalues of $-d\Delta(+BC)$ where the boundary conditions (BC) are $\partial u/\partial n + \theta e u = 0$ in $\partial\Omega$. As in Hale and Rocha [7], we consider the minimum characterization of the first eigenvalue

$$(2.7) \qquad \lambda_1 = \min\left\{d\int_\Omega |\nabla u|^2 + \theta\int_{\partial\Omega} eu^2 : \int_\Omega u^2 = 1\right\}$$

from which we obtain that $0 \leq \lambda_1 \leq \theta\,|\Omega|^{-1}\int_{\partial\Omega} e$.

If $\lambda_1 = \theta\mu_1$, then

$$\mu_1 = \mu_1(d/\theta) \to |\Omega|^{-1}\int_{\partial\Omega} e \qquad \text{as } d/\theta \to \infty.$$

If $\lambda_1 = d\nu_1$, then

$$\nu_1 = \nu_1(\theta/d) \to \nu_{10} > 0 \qquad \text{as } \theta/d \to \infty.$$

where ν_{10} is the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions. Thus, for any $r > 0$, there are $d_0 = d_0(r) > 0$, $\theta_0 = \theta_0(r) > 0$ such that $\lambda_1(d,\theta) > r$ for $d > d_0$, $\theta > \theta_0$.

To estimate λ_2, let $\lambda_2 = d\mu_2$ and obtain

$$\mu_2 = \min\left\{\int_\Omega |\nabla u|^2 + \frac{\theta}{d}\int_{\partial\Omega} eu^2 : \int_\Omega u^2 = 1, \int_\Omega u\phi_1 = 0\right\}$$

where ϕ_1 is the eigenfunction corresponding to λ_1. Then there exist positive constants d_0, μ independent of θ such that $\mu_2 > \mu$ for every $d > d_0$. To see this, consider $\mu_2 = \mu_2(d,\theta)$ and assume the existence of sequences d_j, θ_j such that

$d_j > d_0$ and $\mu_2(d_j,\theta_j) \to 0$. Since $0 \leq \int_\Omega |\nabla\phi_1|^2 \leq \mu_2$, we have $\int_\Omega |\nabla\phi_1|^2 \to 0$. Moreover, denoting by ϕ_2 the eigenfunction corresponding to λ_2, we also have that $\int_\Omega |\nabla\phi_2|^2 \to 0$. Then, from $\int_\Omega \phi_j^2 = 1$, $j = 1,2$, we have that $\phi_j \to |\Omega|^{-1/2}$ contradicting $\int_\Omega \phi_1\phi_2 = 0$. Hence for $d > d_0$ we have the estimate $\lambda_2 > d\mu$ uniformly in θ.

We now prove the following.

Lemma 2.1. Suppose $n = 3$, $\alpha > 3/4$, $X = L^2$ and $X^\alpha = X^\alpha(\theta,d)$ is the fractional power space associated with $-D\Delta$ with boundary conditions (1.2). For any $d_0 > 0$, there is a constant $k(d_0)$ such that for any $d \geq d_0$, $\theta \in [0,\infty)$ and any $u \in X^\alpha(\theta,d)$

$$|u|_{L^\infty} \leq k(d_0)\,|u|_{X^\alpha(\theta,d)}$$

Proof: As for the case $n = 1$, the essential step is to consider the following problem:

$$\Delta u = g \quad \text{for} \quad x \in \Omega \subset \mathbb{R}^n$$

$$B_\theta u = 0 \quad \text{for} \quad x \in \partial\Omega,$$

where $B_\theta u \overset{\text{def}}{=} \partial u/\partial n + \theta u$, $\theta \in [0,\infty)$ and to prove the following uniform regularity estimate:

$$\|u\|_{H^2} \leq M(\|g\|_{L^2} + \|u\|_{L^2}) \tag{2.8}$$

where M is a constant independent of θ, and then use the fact that, for $\theta > 0$, $\|u\|_{L^2} \leq c\,\|g\|_{L^2}$, (Friedman [4, pg. 76]).

Since regularity is a local property, we let $\Sigma\rho_i = 1$ be a partition of unity

subordinate to a neighborhood covering of Ω. Then

$$(2.9)\qquad \|u\|^2_{H^2} = \|\ \Sigma\,\rho_i u\|^2_{H^2} \leq k\,\Sigma\,\|\rho_i u\|^2_{H^2}.$$

If $\rho_i u$ has support in the interior of Ω, then (M. Schechter [13], Lemma 7)

$$\|\rho_i u\|^2_{H^2} \leq C(\|\rho_i u\|^2_{L^2} + \|\Delta\rho_i u\|^2_{L^2}).$$

Since $\Delta\rho_i\, u = \rho_i\,\Delta u$ + derivatives of u of order ≤ 1, we have

$$\|\Delta\rho_i u\|^2_{L^2} \leq 2\|\rho_i\Delta u\|^2_{L^2} + 2\,c_1\,\|u\|^2_{H^1}$$

and from the inequality (L. Nirenberg [12], appendix):

$$(2.10)\qquad \|u\|^2_{H^1} \leq \epsilon'\|u\|^2_{H^2} + k_1(\epsilon')\|u\|^2_{L^2},$$

valid for every ϵ' 0, we finally obtain

$$\|\rho_i u\|^2_{H^2} \leq C(\|\rho_i u\|^2_{L^2} + 2\|\rho_i\Delta u\|^2_{L^2} + 2c_1\epsilon'\|u\|^2_{H^2} + 2c_1\ _1\|u\|^2_{L^2}$$

$$\leq 2C\left[\|\Delta u\|^2_{L^2} + (1/2 + c_1k_1)\|u\|^2_{L^2} + \epsilon' c_1\|u\|^2_{H^2}\right]$$

$$(2.11)\qquad \|\rho_i u\|^2_{H^2} \leq K(\|g\|^2_{L^2} + \|u\|^2_{L^2} + \epsilon\|u\|^2_{H^2}).$$

Now, if the support of $\rho_i u$ contains a piece of $\partial\Omega$, we consider a transformation of variables straightening up the boundary. We let $\partial\Omega_i$ denote the piece of $\partial\Omega$ contained in the support of $v \overset{def}{=} \rho_i u$ and assume without loss of generality that $\partial\Omega_i$ is connected. Let $\psi : \mathbf{R}^n \to \mathbf{R}^n$ be a smooth local change of coordinates mapping the support of v into a ball B_i centered at the origin, and $\partial\Omega_i$ into $B_i \cap T$, where T denotes the hyperplane $T = \{y = (y_1,...,y_n) \in \mathbf{R}^n \,|\, y_n = 0\}$. Under this local change of coordinates, the initial problem is transformed into the following:

$$Lu = g \quad \text{for} \quad y \in B_i \cap \{y_n > 0\}$$

$$B_\theta' u = 0 \quad \text{for} \quad y \in B_i \cap T,$$

where $B_\theta' u = (1-\theta)\,\partial u/\partial y_n - \theta u$, and L is a linear second order strongly elliptic operator with variable coefficients. Let us denote by L_0 the homogeneous operator with constant coefficients which equals the principal part of L at the origin. Then, as in M. Schechter [14], (proof of Lemma 12), we may assume that the change of coordinates ψ (after a rotation) has the form $y_j = x_j$, $j = 1,...,n-1$, $y_n = \phi(x_1,...,x_n)$ such that, at the point $x_0 = \psi^{-1}(0)$, we have $\phi(x_0) = 0$ and also $\partial\phi/\partial x_n = 1$, hence preserving Lebesgue measure. For this change of coordinates, we have $L_0 \equiv \Delta$. Then we consider the problem:

$$\text{(2.12)} \qquad \Delta w = f \quad \text{for} \quad y \in E_+^n = \mathbf{R}^{n-1} \times \mathbf{R}_+$$

$$B_\theta' w = 0 \quad \text{for} \quad y \in \mathbf{R}^{n-1} \times \{0\}.$$

Here, for simplicity, we assume that $n \geq 3$ and introduce the notations $y^* =$

$(y_1,\ldots,y_{n-1},-y_n)$ and ω_n the volume of the unit ball in $\mathbf{R}^n$. Then, from Gilbarg and Trudinger ([6], Chapter 6.7), we can solve (12) in terms of the Green's function:

$$w(x) = \int_{E_+^n} G(x,y)f(y)dy$$

where $G(x,y) = \Gamma(x-y) - \Gamma(x-y^*) + \Sigma(x-y^*)$ with $\Gamma(z) = |z|^{2-n}/n(2-n)\omega_n$ the fundamental solution of Laplace's equation $(n > 2)$, and

$$\Sigma(z) = -2\int_0^\infty e^{-\theta(1-\theta)^{-1}s}\frac{\partial}{\partial y_n}\Gamma(z+e_n s)ds,\ e_n = (0,\ldots,0,1).$$

Since $\theta \in (0,1)$, we have, for $z \in E_+^n$,

$$|\Sigma(z)| \leqslant 2\int_0^\infty |\frac{\partial}{\partial y_n}\Gamma(z+e_n s)|\,ds = 2\Gamma(z),$$

$$|w(x)| \leqslant \int_{E_+^n} H(x,y)\,|f(y)|\,dy,$$

where $H(x,y) = \Gamma(x-y) + 3\Gamma(x-y^*)$. Then, as in Agnon, Douglis and Nirenberg [1], we can extend these kernels to $\mathbf{R}^n$ as odd in x_n, and apply the Calderon-Zygmund theorem, obtaining

$$\|w\|_{L^2}^2 \leqslant N_1\|f\|_{L^2}^2.$$

One can do the same after differentiating w twice and passing the derivatives to the kernel G, obtaining then

(2.13) $$\|w\|^2_{H^2} \leq N_2 \|f\|^2_{L^2}.$$

where the constants N_i do not depend upon θ. Then, since $L_0 v = Lv + (L_0 - L)v$, we have

$$\|L_0 v\|^2_{L^2} \leq 2\|Lv\|^2_{L^2} + 2\|(L_0 - L)v\|^2_{L^2},$$

and again, by (10),

$$\|L_0 v\|^2_{L^2} \leq 2\|Lv\|^2_{L^2} + \delta'\|v\|^2_{H^2} + N'\|v\|^2_{L^2}.$$

Thus, from (2.13), considering a partition of unity sufficiently small so that $\delta' \leq (2N_2)^{-1}$, we obtain

$$\|L_0 v\|^2_{L^2} \leq 2\|Lv\|^2_{L^2} + \delta' N_2 \|L_0 v\|^2_{L^2} + N'\|v\|^2_{L^2},$$

$$\|L_0 v\|^2_{L^2} \leq k_2(\|Lv\|^2_{L^2} + \|v\|^2_{L^2}).$$

So, again by (2.13), we see that

$$\|\rho_i u\|^2_{H^2} \leq C(\|\rho_i u\|^2_{L^2} + \|L\rho_i u\|^2_{L^2}).$$

As before, we can now obtain

$$\|\rho_i u\|^2_{H^2} \leq K(\|g\|^2_{L^2} + \|u\|^2_{L^2} + \epsilon\|u\|^2_{H^2})$$

and, from (2.9), we have

$$\|u\|^2_{H^2} \leq \bar{K}(\|g\|^2_{L^2} + \|u\|^2_{L^2} + \epsilon\|u\|^2_{H^2}) .$$

Then, choosing ϵ sufficiently small, we finally obtain the desired estimate (2.8). This completes the proof of the lemma.

3. An example.

In this section, we discuss the situation in which u in (1.1), (1.2) is a scalar, $\Omega = (0,1)$ and f(u) is a cubic. It is convenient in the computations to replace the parameters (d,θ) by $(d^2, \theta/(1-\theta)d)$. The example to be considered is

$$(3.1)_a \qquad \begin{aligned} &u_t = d^2 u_{xx} + f_a(u) && && x \in (0,1) \\ &(1-\theta)du_x = \theta u && \text{at} && x = 0 \\ &(1-\theta)du_x = -\theta u && \text{at} && x = 1 \end{aligned}$$

where $d \in (0,\infty)$ and $\theta \in [0,1]$ and

$$(3.2) \qquad f_a(u) = u(1-u)(u-a), \qquad a \in [-1,1]$$

Since (3.1) is a gradient system every solution approaches an equilibrium solution. In the rescaled variables $x = dy$, these solutions satisfy

$$(3.3)_a \qquad u_{yy} + f_a(u) = 0, \quad y \in (0,d^{-1})$$

and boundary conditions:

$$(3.4) \qquad \begin{aligned} &(1-\theta)u_y = \theta u && \text{at} && y = 0 \\ &(1-\theta)u_y = -\theta u && \text{at} && y = d^{-1}. \end{aligned}$$

Since the set of equilibrium solutions is bounded, there is a compact attractor $\mathcal{B}_{d,\theta}$ for every $d > 0$, $\theta \in [0,1]$ (see, for example Henry [10] or Hale [9]).

If $W^u(\phi)$, $W^s(\phi)$ are the stable and unstable sets for an equilibrium solution then a recent result of Henry [11] shows that $W^u(\phi)$ is transversal to $W^s(\psi)$ for all equilibrium solutions ϕ,ψ. This implies that the flow defined by (3.1) is

structurally stable if and only if the equilibrium solutions are hyperbolic; that is, if and only if each equilibrium solution has the property that its linear variational equation has nonzero eigenvalues. This implies that the curves in the (d,θ)-plane which correspond to bifurcation points of the flow must be either primary bifurcations from an equilibrium or saddle-node bifurcations of equilibria. The purpose of this section is to discuss these curves for $(3.3)_a$ for values of $a \in [-1,1]$. For a = -1 we prove the following result for the case a = -1; that is $f(u) = u - u^3$.

Theorem 3.1. Let $s_j \subset (0,\infty) \times [0,1]$ be the structurally stable regions for $(3.1)_{-1}$ which consists of exactly $2j+1$ hyperbolic equilibrium points. Then the following relations hold:

1) S_j has only one connected component,
2) S_0, S_1 are unbounded, S_j is bounded for $j \geq 2$,
3) $S_0 \cap \{\theta = 0\} = \phi$, $S_0 \cap \{\theta = 1\} \neq \phi$
4) $S_j \cap \{\theta = 0\} \neq \phi$, $S_j \cap \{\theta = 1\} \neq \phi$, $\forall j \geq 1$,

   and, for each integer k,

   $$(C\ell\ U_{j \geq k+1} S_j) \cap \{\theta = 0\} = (C\ell\ U_{j \geq k} S_j) \cap \{\theta = 1\} = [0,d_k],$$

   where $d_k = (k\pi)^{-1}$.
5) ∂S_j are smooth C^1-curves nonincreasing in θ. These curves are nonintersecting in $(0,\infty) \times (0,1]$.

Before proving this result, we make the following remarks.

Remark 3.2. Suppose d_1 is as in property 4) and $d_0 > d_1$ is fixed. From properties 2) and 5), if one studies the attractor $\mathcal{B}_{d,\theta}$ as a function of θ for a fixed

$d > d_0$, then one must go from a situation of three equilibrium points at $\theta = 0$ to one equilibrium point at $\theta = 1$. Furthermore, according to property 5), there is only one point θ at which a bifurcation occurs. This d_0 provides a good estimate of the d_0 occurring in Theorem 1.1.

Remark 3.3. Properties 1) and 4) imply that one can find a homotopy from any structurally stable system with Neumann conditions to a structurally stable one with Dirichlet conditions. The case with $a \in (0,1/2)$ was considered by Gardner [5] where he shows the existence of such a homotopy for the case with three equilibria. We will see later that for a $[1/2,1]$ no such homotopy exists.

Proof of Theorem 3.1. Let $L_\pm$ be the lines in the (u,u_y)-plane defined by $L_\pm = \{(u,u_y) : (1-\theta)u_y = \pm\theta u\}$. Let $u = u(y,u_0)$ be a solution of $(3.3)_{-1}$, where u_0 corresponds to the maximum value of u. If this maximum occurs at $y = \tau$, then $u(\tau,u_0) = u_0$ and $u_y(\tau,u_0) = 0$. We define the "time map" T to be $T(u_0) = \tau$. From this time map, the existence of solutions of $(3.3)_{-1}$ can be inferred. Such a solution exists if and only if there exists a $u_0 \in (0,1)$ for which $T(u_0) = (2d)^{-1}$. Introducing the polar coordinates $u = r \cos s$, $u_y = -r \sin s$ in $(3.3)_{-1}$, one can show that $s = s(y,u_0)$ satisfies the differential equation:

$$s_y = \sin^2 s + (1 - r^2\cos^2 s)\cos^2 s, \qquad y \in [0,d^{-1}],$$

$$s(0) = -\phi; \quad \phi \stackrel{def}{=} \text{arctg}\ \theta/(1-\theta) \qquad (0,\pi/2).$$

From this equation we determine the following expression for the time map:

$$(3.5) \qquad T(u_0) = \int_0^{\phi} [\sin^2 s + (1 - r^2\cos^2 s)\cos^2 s]^{-1} ds$$

where $r = r(s,u_0)$. Then, as in Hale and Rocha [7], we can prove that the time map is a monotone increasing function of $u_0 \quad (0,1)$. In fact, differentiating (3.5), we have:

$$(3.6) \qquad T'(u_0) = 2 \int_0^{\phi} [\sin^2 s + (1 - r^2\cos^2 s)\cos^2 s]^{-2}\, r \,\frac{\partial r}{\partial u_0}\, \cos^4 s \, ds$$

and $\partial r/\partial u_0 > 0$ and $\phi \quad (0,\pi/2)$ imply that $T'(u_0) > 0$ for $u_0 \quad (0,1)$. Thus, as in Chafee and Infante [3], the bifurcation of equilibrium solutions can only occur at the origin. This will happen for the values of $d = d(\theta)$ corresponding to the zero eigenvalue for the linearized problem:

$$(3.7) \qquad u_{yy} + u = 0, \quad y \quad (0,d^{-1}),$$

and boundary conditions (3.4). Then, we will have $u = A \cos y + B \sin y$, and the boundary conditions will be satisfied if and only if:

$$(3.8) \qquad d = \left[\text{arc cotg}\, \frac{1}{2}\left(\frac{1-\theta}{\theta} - \frac{\theta}{1-\theta}\right)\right]$$

This provides an expression for the curves ∂S_j referred to in property 5) of the Theorem 3.1. Moreover, one also concludes that the d_k in property 4) are given by $d_k = (k\pi)^{-1}$. This completes the proof of the theorem.

Figure 3.a presents the sets S_j, and Figure 3.6 the bifurcation diagram for a fixed value of $\theta \quad (0,1)$.

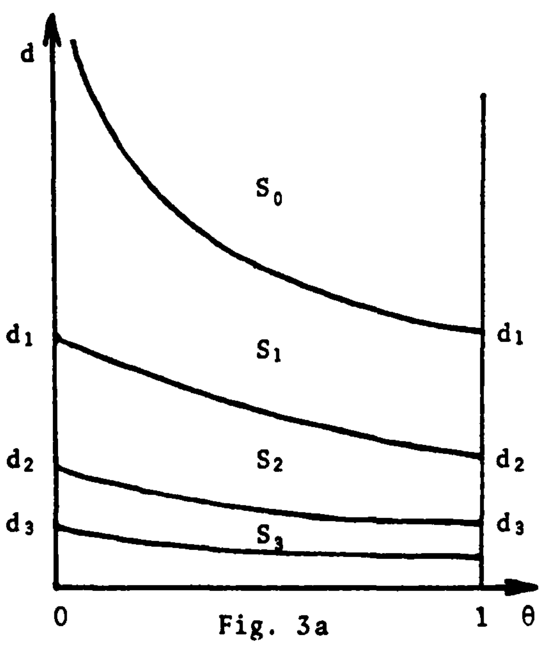

Fig. 3a

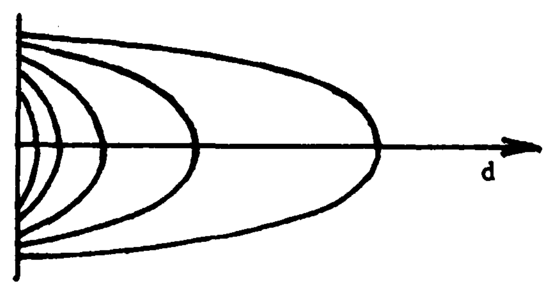

Fig. 3b

In the following, we consider the time map and its derivatives defined at $u_0 = 0$ by continuity. Then, one can easily check that $T(0) = \phi$, $T'(0) = 0$ and $T''(0) > 0$ for $\theta \in (0,1]$. For $a \neq -1$ the problem becomes much more difficult because the expressions for the derivatives of the time map are more complicated. Nevertheless, we can prove the following:

<u>Theorem 3.4. For every</u> D <u>sufficiently large, let</u> $S_j \subset (0,D] \times [0,1]$ <u>denote the structurally stable regions for</u> $(3.1)_a$ <u>consisting of exactly</u> $2j+1$ <u>hyperbolic equilibrium points. Then there exists a</u> $c \in (0,1)$ <u>such that for all</u> $a \in (-1,-1+c)$ <u>the following hold:</u>

1) S_j <u>has only one connected component if</u> $j = 2k$, $k \geq 0$.

2) S_j <u>has exactly two connected components if</u> $j = 2k+1$, $k \geq 0$.

<u>Moreover, the relations</u> 3) <u>to</u> 5) <u>of theorem 3.1 still hold</u>.

To prove this, we introduce in the time map the dependence on a, $T = T(u_0,a)$:

(3.9) $$T(u_0,a) = \int_0^{\phi} \{\sin^2 s + [-a + (1+a)r \cos - r^2\cos^2 s]\cos^2 s\}^{-1} ds$$

From the remark before the statement of the theorem, we know that $\partial T/\partial u_0(0,-1) = 0$ and $\partial^2 T/\partial u_0^2(0,-1) > 0$ for $\theta \in (0,1]$. In the same way, one can verify that $\partial^2 T/\partial a \partial u_0(0,-1) < 0$ for $\theta \in (0,1]$. Then, for any $\delta > 0$, we can find an $\epsilon > 0$ such that, for θ $[\delta,1]$, we have $\partial^2 T/\partial u_0^2(0,-1) > \epsilon$ and $\partial^2 T/\partial a \partial u_0(0,-1) < -\epsilon$. Hence, for $\theta \in [\delta,1]$, the changes introduced in the time map as $a > -1$ are very simple, and we can find a $c \in (0,1]$ such that for all $a \in (-1,-1+c)$ the time map has a unique extremum at $\bar{u}_0 \in (0,1)$, which is a minimum. This gives us the shape of the first bifurcation curve in Figure 4.a, showing what is usually called a transcritical bifurcation at the origin. A simple analysis of the phase plane shows that only

the odd bifurcations at the origin will be transcritical, the even ones being supercritical. This observation takes care of the curves ∂S_j in the region $[0,D] \times (\delta,1]$. For the region $(0,D) \times (\delta,1]$, we start by observing that if $\delta = \delta(D)$ is small enough this region always contains at least three hyperbolic equilibria, thus, the first bifurcation of the origin is excluded. Then, one needs only to consider the solutions arising from the second, third, etc., bifurcations. If we define

$$U_j(u_0,a) \stackrel{\text{def}}{=} \int_{-\phi}^{\phi+(j-1)\pi} (\sin^2 s + [-a + (1+a)r\cos s - r^2\cos^2 s]\cos^2 s)^{-1} ds,$$

$$j = 1,2,\ldots$$

then U_j represents the value of y at which the solution of $(3.3)_a$ satisfying the initial condition in (3.4) and having at the first maximum the value u_0, satisfies the final condition in (3.4) after passing through j extrema. Note the relation with the time map: $U_1(u_0,a) = 2T(u_0,a)$. One can clearly use U_j to determine the existence of solutions of $(3.3)_a$ in the same way as the time map was used. As before, we can verify now that, for all $\theta \in [0,1]$, $\partial^2 U_j/\partial u_0^2(0,-1) > 0$ for $j \geq 2$, $\partial U_j/\partial u_0(0,a) = 0$ for $j = 2k$ and all $a \geq -1$, and $\partial^2 U_j/\partial a \partial u_0(0,-1) < 0$ for $j = 2k+1$, $k = 1,2,\ldots$. Hence, the changes introduced in U_j, $j \geq 2$, as $a > -1$ are very simple and again we can find a $c \in (0,1]$ such that, for all $a \in (-1,-1+c)$, U_j, $j \geq 2$, has a unique extremum which is a minimum. This minimum occurs at the origin if $j = 2k$, and at $\bar{u}_j \in (0,1)$ if $j = 2k+1$, for $k = 1,2,\ldots$. This justifies the bifurcation diagram presented in Figure 4.a, and concludes the proof of the theorem. In Figure 4.b, we present the sets S_j as obtained from the theorem. It turns out that, if we consider the linearized problem $u_{yy} - au = 0$ and compare with (3.7), we obtain an expression for the curves ∂S_j corresponding to the bifurcations at the origin if we multiply (3.8) by the factor $(-a)^{1/2}$ for $a \in [-1,0)$.

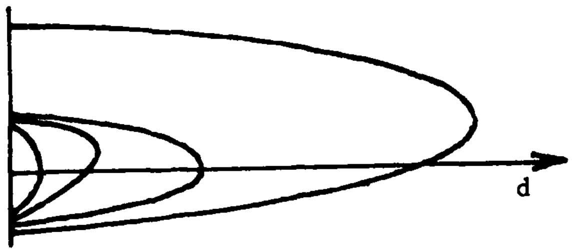

Figure 4.a

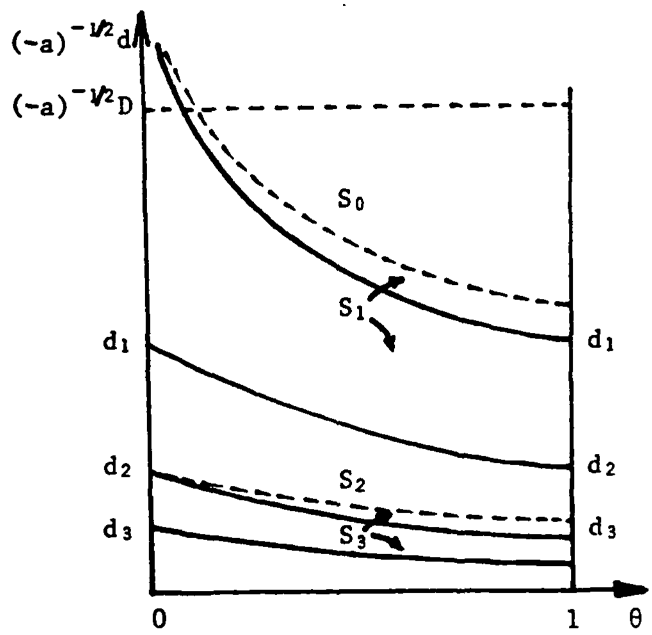

Figure 4.b

If one considers the results obtained by Smoller and Wasserman [15] for the cases of Dirichlet and Neumann boundary conditions, the results of this theorem are not surprising. Moreover, numerical tests indicate that these results seem to hold for all a in [-1,0); thus, the maximum value of c in theorem 3.4 being possibly 1.

For a = 0, the problem is degenerate and does not have any structurally stable regions. It becomes then very interesting to make the same study for $a \in (0,1)$. This problem is as difficult as the previous one for $a \in [-1.0)$ for the same reasons. Therefore, we concentrate on qualitative information. Considering the phase diagram corresponding to the equation $(3.3)_a$, one notices that there is a qualitative change as a crosses the value 1/2. In fact, for $a \in (0,1/2)$, this diagram contains a homoclinic orbit to the point $(u,u_y) = (0,0)$; at a = 1/2, it contains two heteroclinic orbits to the points (0,0) and (1,0), and, for $a \in (1/2,1)$, the diagram

has an orbit homoclinic to the point (1,0). This qualitatiave change reflects on the structurally stable regions for $(3.1)_a$. Let us define a function $\gamma = \gamma(a)$ in the following way. For $a \in (0,1/2]$, $\gamma(a)$ is the value of θ in $L_\pm$ corresponding to the angle of the tangents to the separatrices at the origin in the phase diagram. A simple computation yields $\gamma(a) = (1 + a^{-1/2})^{-1}$. For $a \in (1/2,1)$ we define $\gamma(a)$ as the value of θ in $L_\pm$ corresponding to the tangents to the homoclinic orbit passing through $u = 1$. This function γ is a continuous function satisfying $0 < \gamma(a) \leq (1+\sqrt{2})^{-1}$ for $a \in (0,1)$. Then, if again we let S_j denote the structurally stable regions for $(3.1)_a$ corresponding to $2j+1$ equilibria, we have:

<u>Theorem 3.5</u>: <u>For</u> $a \in (0,1)$ <u>the following holds</u>:

1) <u>If</u> $a < 1/2$, <u>there exist positive constants</u> δ <u>and</u> D <u>such that</u>

   $(0,\delta) \times [\gamma(a),1] \subset S_1$,

   $[D,\infty) \times [\gamma(a),1] \subset S_0$.

2) <u>If</u> $a \geq 1/2$, <u>we have</u>

   $(0,\infty) \times [\gamma(a),1] \subset S_0$.

To prove this, we consider the effect that the above observation about the phase diagram for $(3.1)_a$ has upon the time map as defined by (3.9). If, for a (0,1/2), we denote by α the smallest positive root of $\int_0^u f_a(s)ds = 0$, then the point $(\alpha,0)$ of the phase diagram corresponds to the intersection of the homoclinic orbit through the origin with the positive u-axis. Then, for $\theta \geq \gamma(a)$, the time map $T(\cdot,a) : (\alpha,1) \to (0,\infty)$ is continuously differentiable and unbounded as $u_0 \to \alpha$ or 1. Moreover, one can easily check that $\partial T/\partial u_0(\cdot,a) : (\alpha,1) \to \mathbf{R}$ is positive as u_0 approaches 1 and negative as u_0 approaches α. This implies the existence of a minimum value $p_0 > 0$ for $T(\cdot,a)$, and also a maximum value $p_1 \geq p_0$ for its

extrema. Then, if we take $D > (2p_0)^{-1}$, for $d \geq D$ there will be no nonconstant solutions for $(3.1)_a$, and taking $\delta = (2p_1)^{-1}$ there will be exactly two nonconstant hyperbolic solutions for $d < \delta$. Moreover, for $\theta \in (0,1]$ and all $a \in (0,1)$, zero is the only constant solution, being always hyperbolic. This completes the proof of part 1) in the theorem. Part 2) will follow from the observation that for a $[1/2,1)$ and $\theta \geq \gamma(a)$ there are no nonconstant solutions of $(3.1)_a$. In Figures 5.a and 5.b, we present our conjecture for the shapes of S_j in the cases of $a \in (0,1/2)$ and $(1/2,1)$, respectively.

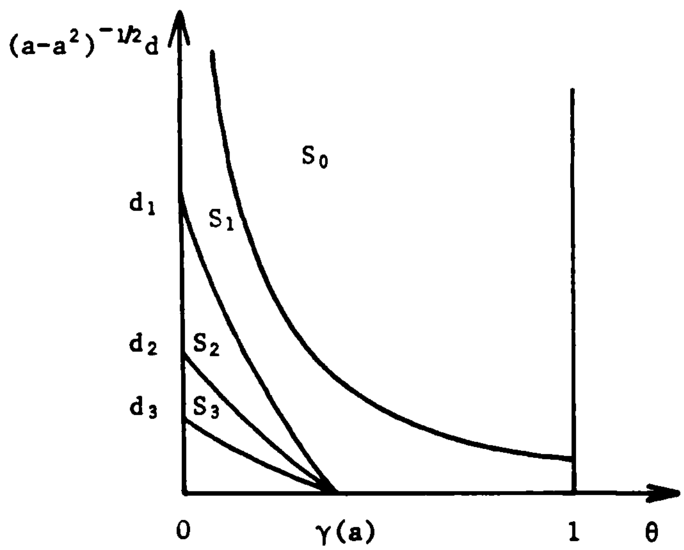

Figure 5.a

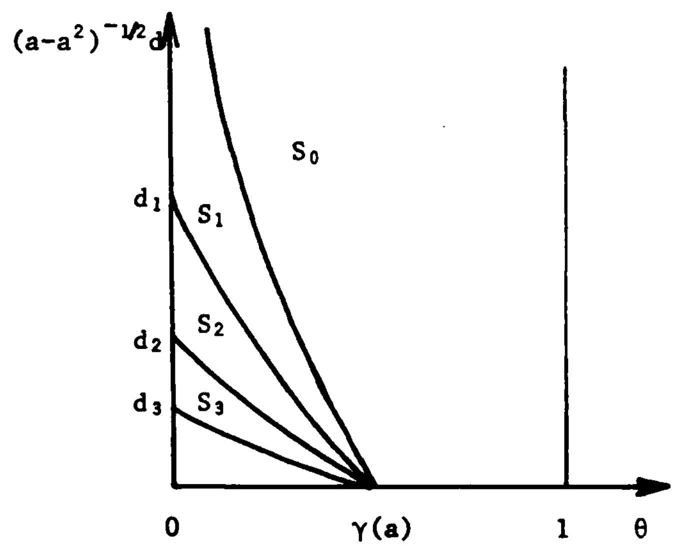

Figure 5.b

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Remark 3.6. It is clear that part 2) of the previous theorem presents the existence of a homotopy from a structurally stable system with Neumann boundary conditions, which must have at least three solutions, to a structurally stable system with Dirichlet boundary conidtions, which must have only one solution. The above example also makes clear what one should do to create such qualitative phenomena as the alternative in Theorem 3.5, in systems corresponding to more general functions f. Finally, in this example the curves ∂S_j in theorems 3.1 and 3.4 were constituted only by codimension one bifurcations. It is possible to create examples in which these lines intersect, presenting higher codimension bifurcations. For instance, take an example for which the time map at some θ_0 has two minima which are equal. Then, for this example there would be a $d_0 > 0$ such that (d_0,θ_0) would correspond to a codimension two bifurcation.

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